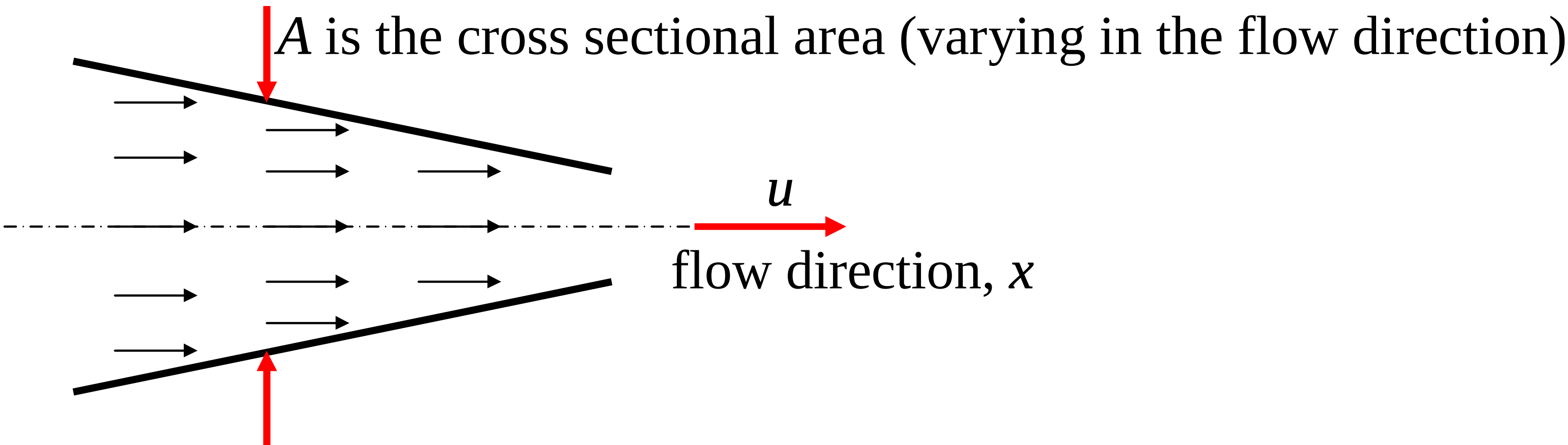
CP502
Advanced Fluid Mechanics

Compressible Flow

Lectures 4

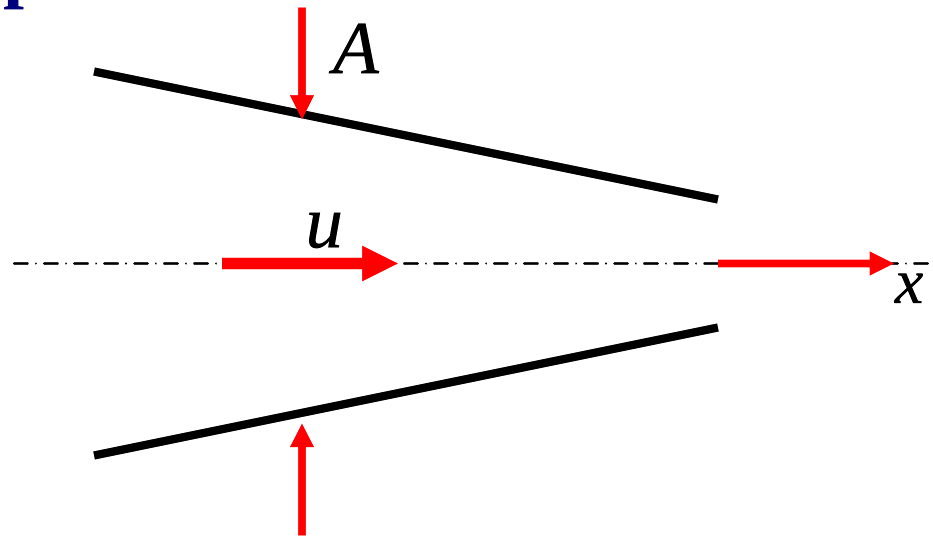
Steady, quasi one-dimensional, isentropic compressible flow of an ideal gas in a variable area duct

Quasi one-dimensional flow in a variable area duct



velocity (u) is uniform across the cross-sectional area (A) and varying only in the flow direction x.

Steady, compressible flow in a variable area duct



A – cross sectional area (varying in the flow direction)

u – velocity uniform across A (varying in the flow direction)

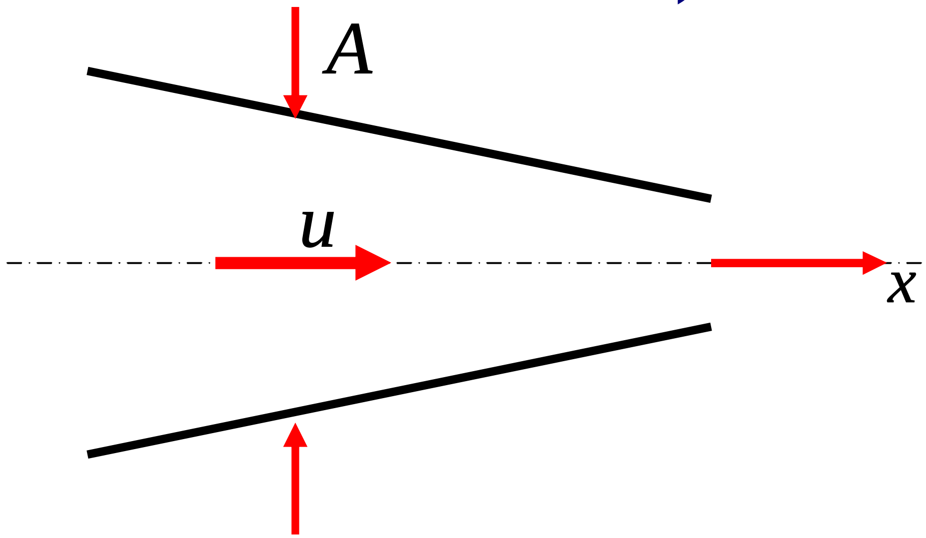
ρ – density uniform across A (varying in the flow direction)

Mass flow rate is a constant for steady flow

$$\dot{m} = A\rho u = \text{constant} \qquad (1)$$

$$d(A\rho u) = 0 \qquad (2.1)$$

Isentropic (adiabatic and inviscid) flow of an ideal gas



p – pressure varies in the flow direction

ρ – density varies in the flow direction (compressible)

T – temperature varies in the flow direction (adiabatic)

Ideal gas satisfies $p = \rho RT$ (2)

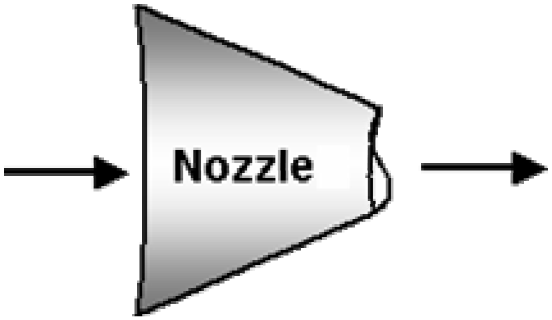
Isentropic flow of an ideal gas satisfies $p \propto \rho^{\gamma}$

$$p = k\rho^{\gamma} \quad (3)$$

(where k is a constant)

Quiz:

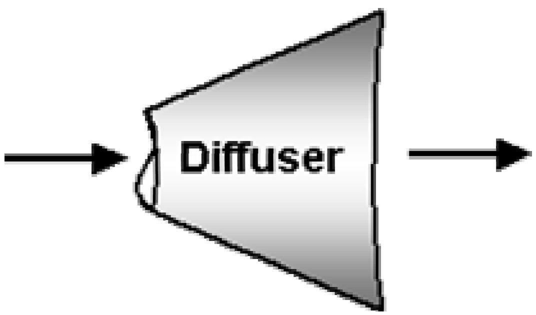
For incompressible flow (with constant density)



u increases or decreases?

p increases or decreases?

T increases or decreases?



u increases or decreases?

p increases or decreases?

T increases or decreases?

Problem 1 from Problem Set #2 in Compressible Fluid Flow:

Consider the following equations governing the steady, isentropic (adiabatic and inviscid), quasi one-dimensional, compressible flow:

$$d(A\rho u) = 0 \quad \text{——} (2.1)$$

$$dh + udu = 0 \quad \text{——} (2.2)$$

$$dp + \rho udu = 0 \quad \text{——} (2.3)$$

If the given flow experiences negligible changes in its density (that is, if the flow is assumed to be **incompressible**) then show that the given flow is described by the following equations:

$$Au = \text{constant}; \qquad h + \frac{u^2}{2} = \text{constant}; \qquad \frac{p}{\rho} + \frac{u^2}{2} = \text{constant}$$

Hence, show that a diverging duct compresses, heats and slows down a steady, inviscid, incompressible flow through it.

And, show that a converging duct expands, cools and speeds up a steady, inviscid, incompressible flow through it.

For incompressible flow, **ρ = constant**

Integrating $d(A\rho u) = 0$ gives $Au = \text{constant}$ ——— (4)

Integrating $dh + udu = 0$ gives $h + \dfrac{u^2}{2} = \text{constant}$ ——— (5)

Integrating $dp + \rho udu = 0$ gives $\dfrac{p}{\rho} + \dfrac{u^2}{2} = \text{constant}$ ——— (6)

Consider a diverging duct:

A increases ⇒ u decreases to satisfy (4) ⇒ h increases to satisfy (5)

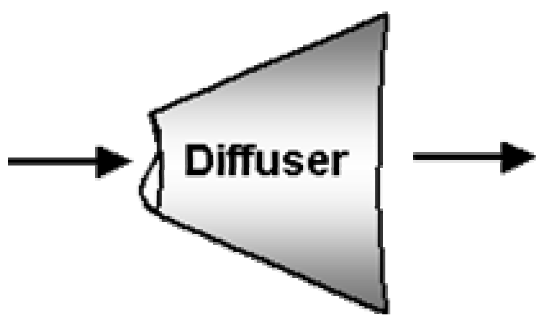
u decreases ⇒ p increases to satisfy (6)

h increases ⇒ T increases in an ideal gas

Fluid slows down

Fluid is compressed

Fluid is heated up



For incompressible flow, **ρ = constant**

Integrating $d(A\rho u) = 0$ gives $Au = \text{constant}$ (4)

Integrating $dh + udu = 0$ gives $h + \frac{u^2}{2} = \text{constant}$ (5)

Integrating $dp + \rho udu = 0$ gives $\frac{p}{\rho} + \frac{u^2}{2} = \text{constant}$ (6)

Consider a converging duct:

A decreases → *u* increases to satisfy (4) → *h* decreases to satisfy (5)

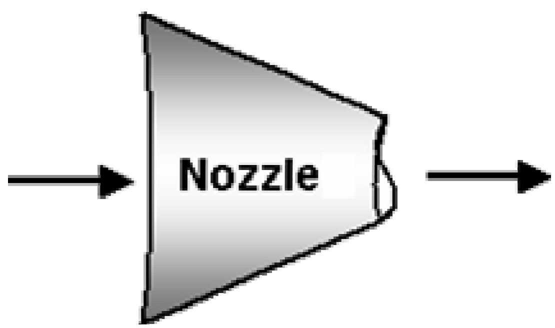
p decreases to satisfy (6)

T decreases in an ideal gas

Fluid speeds up

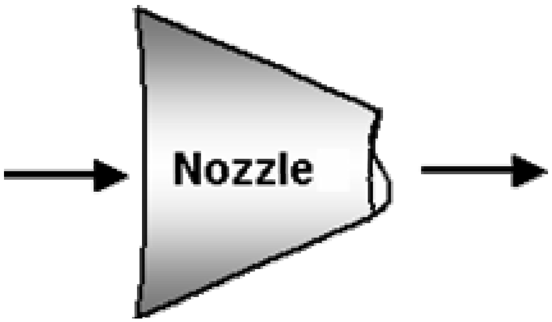
Fluid expands

Fluid is cooled down



Summary:

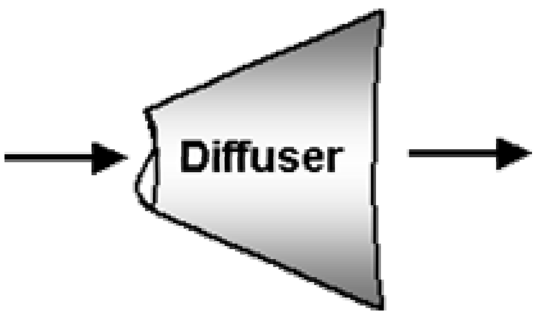
For incompressible flow (with constant density)



u increases

p decreases

T decreases



u decreases

p increases

T increases

Will it be the same for compressible flow?

Problem 2 from Problem Set #2 in Compressible Fluid Flow:

Using the momentum balance (Equation (2.3)) describing the steady, inviscid, quasi one-dimensional flow, explain the following:

(i) Velocity increase is accompanied by pressure decrease

(ii) Velocity decrease is accompanied by pressure increase

Note that the above nature of a flow is common for both compressible and incompressible flows.

Momentum balance describing the steady, inviscid, quasi one-dimensional flow:

$$dp + \rho u du = 0 \qquad (2.3)$$

$$\frac{dp}{du} = -\rho u < 0$$

Therefore,

(i) velocity increase is accompanied by pressure decrease

and

(ii) velocity decrease is accompanied by pressure increase

True for both compressible and incompressible flows.

Problem 3 from Problem Set #2 in Compressible Fluid Flow:

Show that the steady, isentropic, quasi one-dimensional, compressible flow through a varying area duct shall be governed by the following equation:

$$\frac{du}{dA} = \frac{u}{A(M^2 - 1)} \qquad (2.4)$$

a) For a compressible flow at subsonic speeds, show that decreasing duct area increases the flow speed and that a increasing duct area decreases the flow speed.

b) For a compressible flow at supersonic speeds, show that decreasing duct area decreases the flow speed and that a increasing duct area increases the flow speed.

$$\frac{du}{dA} = \frac{u}{A(M^2 - 1)}$$ must be proven

Start with the mass balance describing the steady, isentropic, quasi one-dimensional, compressible flow through a varying area duct:

$$d(A\rho u) = 0 \qquad (2.1)$$

$$A\rho du + \rho u dA + A u d\rho = 0$$

Dividing the above by $A\rho u$, we get

$$\frac{du}{u} + \frac{dA}{A} + \frac{d\rho}{\rho} = 0 \qquad (7)$$

In (7), $d\rho$ must be replaced by either dA or du or both.

Use the equation satisfying isentropic flow of an ideal gas:

$$p = k\rho^{\gamma} \qquad (3) \qquad \text{(where } k \text{ is a constant)}$$

$$dp = \gamma k \rho^{\gamma-1} d\rho \qquad (8)$$

Dividing (8) by (3), we get $\dfrac{dp}{p} = \gamma \dfrac{d\rho}{\rho}$

Using the momentum balance describing the steady, inviscid, quasi one-dimensional flow, $dp + \rho u du = 0$, and the ideal gas equation $p = \rho RT$, the above expression could be rewritten as follows:

$$\frac{d\rho}{\rho} = \frac{1}{\gamma}\frac{dp}{p} = -\frac{1}{\gamma}\frac{\rho u du}{\rho RT} = -\frac{u du}{\gamma RT} \qquad (9)$$

Substituting (9) in (7), we get the following:

$$\frac{du}{u} + \frac{dA}{A} - \frac{du}{\gamma RT} = 0$$

$$\Downarrow$$

$$\left(1 - \frac{u^2}{\gamma RT}\right)\frac{du}{u} + \frac{dA}{A} = 0$$

Using the definition of *M* in the above, we get

$$\frac{du}{dA} = \frac{u}{A(M^2 - 1)} \qquad (2.4)$$

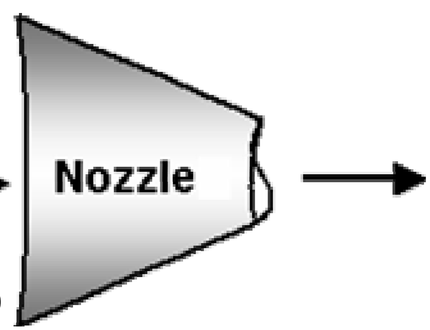
a) For a compressible flow at subsonic speeds, show that decreasing duct area increases the flow speed and that a increasing duct area decreases the flow speed.

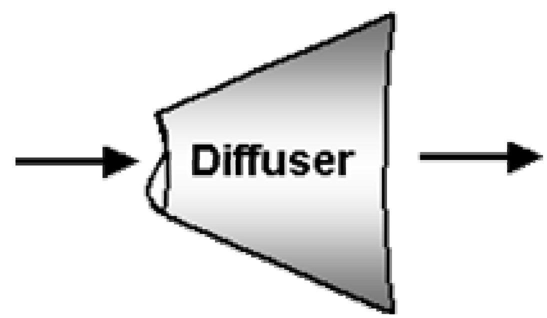
Start with

$$\frac{du}{dA} = \frac{u}{A(M^2 - 1)} \qquad (2.4)$$

At subsonic speed, $M < 1$ and therefore $du/dA < 0$

Decreasing duct area gives $dA < 0$ and therefore $du > 0$ (flow speed increases)	Increasing duct area gives $dA > 0$ and therefore $du < 0$ (flow speed decreases)





b) For a compressible flow at supersonic speeds, show that decreasing duct area decreases the flow speed and that a increasing duct area increases the flow speed.

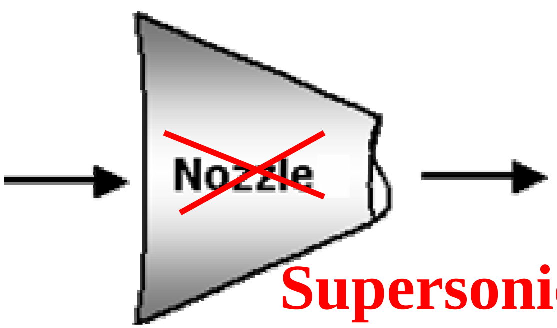
Start with

$$\frac{du}{dA} = \frac{u}{A(M^2 - 1)} \qquad (2.4)$$

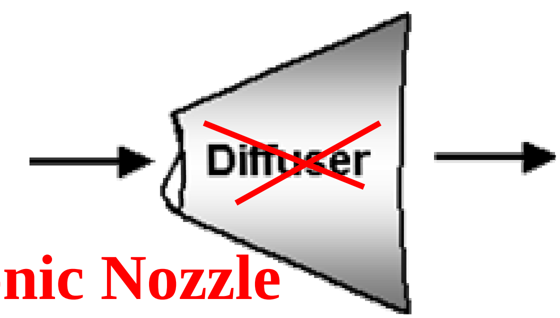
At supersonic speed, $M > 1$ and therefore $du/dA > 0$

Decreasing duct area gives $dA < 0$ and therefore $du < 0$ (flow speed **decreases**)

Increasing duct area gives $dA > 0$ and therefore $du > 0$ (flow speed **increases**)

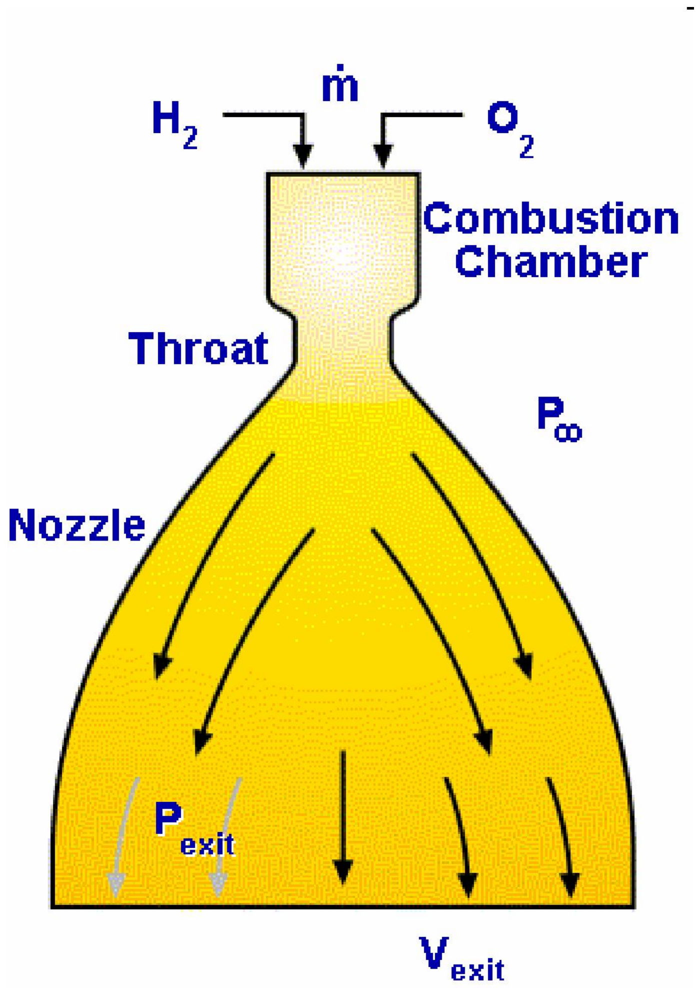


Supersonic Diffuser



Supersonic Nozzle

Rocket Nozzle



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15 Dec 2010

Problem 4 from Problem Set #2 in Compressible Fluid Flow:

Show that, at $M = 1$, du/u can be finite only if the area of the duct is at its minimum. That is, the sonic speed can be attained only at the throat of the duct.

Note: It is, however, not necessary for M to be always 1 at the throat. If M is not 1 at the throat then the velocity attains a maximum or minimum there, depending on whether the flow is subsonic or supersonic.

Start with

$$\frac{du}{dA} = \frac{u}{A(M^2 - 1)} \qquad (2.4)$$

Rearranging (2.4) gives $\left(M^2 - 1\right)\dfrac{du}{u} = \dfrac{dA}{A}$

When $M = 1$, the above equation gives the following alternative solutions:

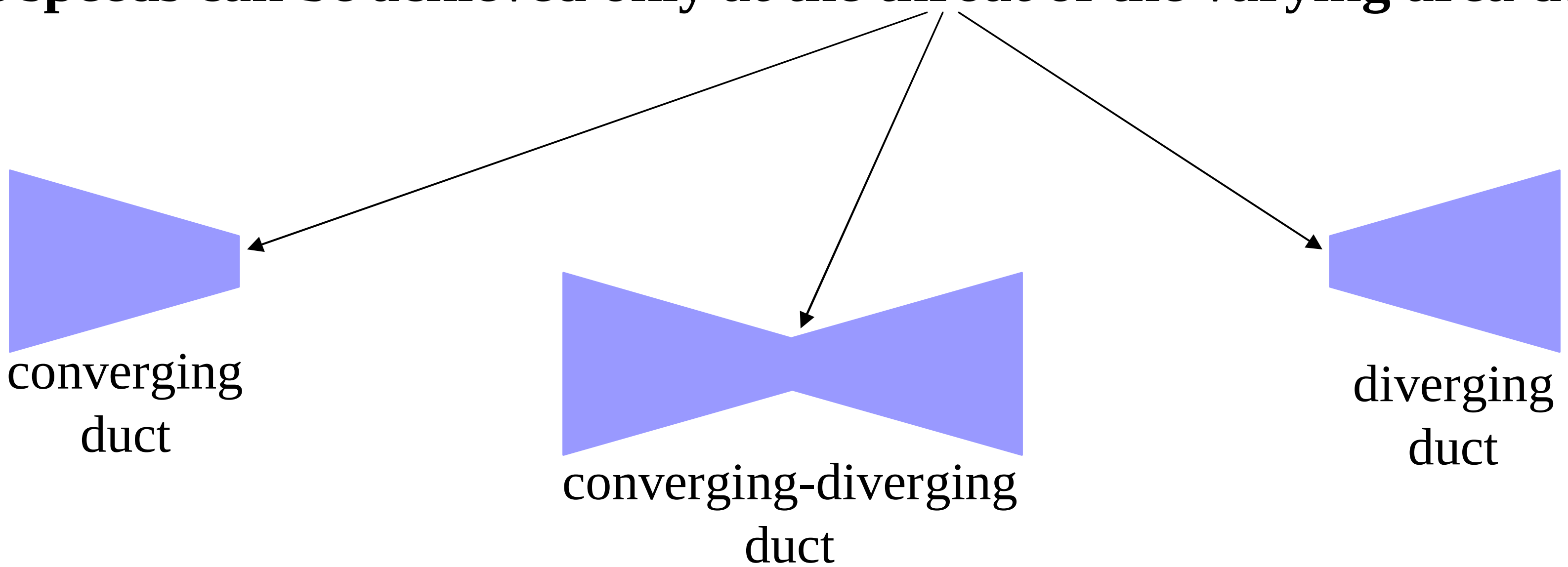
- either du/u is infinity giving finite value for dA/A
- or $dA/A = 0$ giving finite value for du/u

Therefore, for *du/u* to be finite, *dA/A* = 0 at *M* = 1.

dA = 0 at *M* = 1 since *A* is finite

A is at its minimum at *M* = 1

Sonic speeds can be achieved only at the throat of the varying area duct.



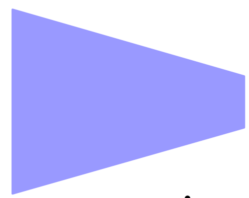
Note that it is not necessary for M to be always 1 at the throat.

If ***M* is not 1** at the throat, $(M^2 - 1)\dfrac{du}{u} = \dfrac{dA}{A}$ gives $du = 0$ at the throat.

u is at its minimum or maximum

at the throat if $M \neq 1$ at the throat.

If the flow is subsonic, u is at its maximum at the throat of a converging duct.



converging
duct

If the flow is supersonic, u is at its minimum at the throat of a converging duct.

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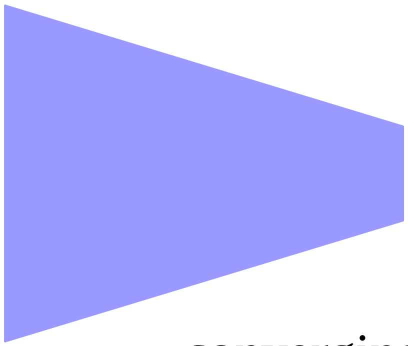
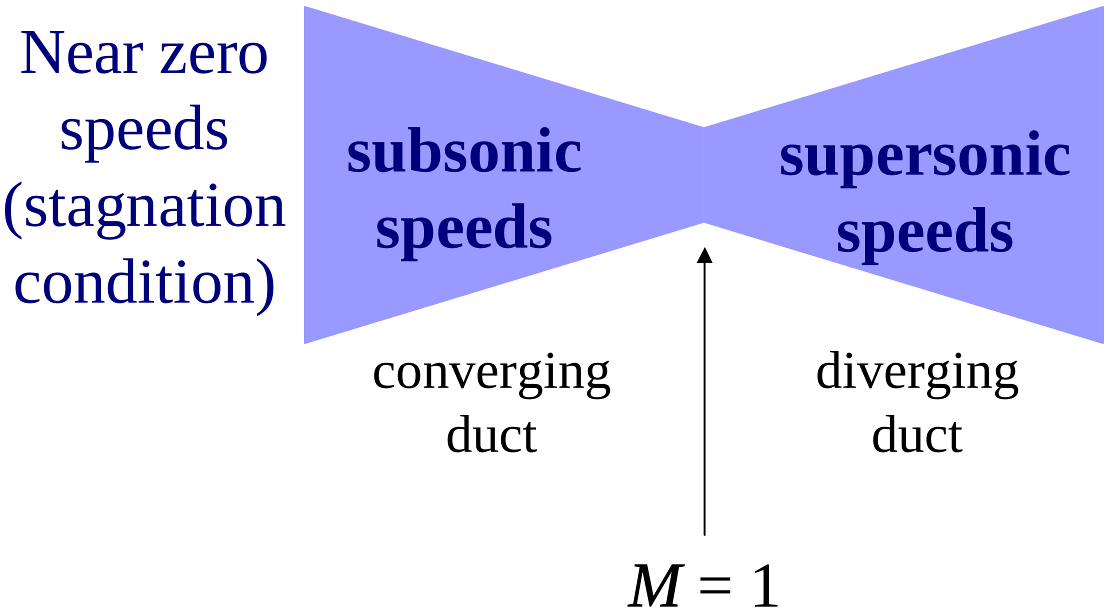
Can we achieve supersonic speeds from near zero speeds using a converging duct?



Can reach sonic speeds at the throat of a converging duct.

And, that is all could be achieved.

How can we achieve supersonic speeds from near zero speeds using a varying area duct?



A converging-diverging duct is necessary to reach supersonic speeds starting from stagnation condition.

And, a diverging-converging duct is necessary to slow down from supersonic speeds to stagnation condition.