APPENDIX F
Thermal Energy Equation

F.1 RECTANGULAR COORDINATES

The following form of the thermal energy equation in rectangular coordinates allows for nonconstant physical properties, energy generation, and conversion of mechanical to internal energy through viscous dissipation, which is expressed in terms of unspecified viscous stress–tensor components τ_{ij}:

$$\begin{aligned}\rho C_v \frac{\partial T}{\partial t} + \rho C_v u_x \frac{\partial T}{\partial x} + \rho C_v u_y \frac{\partial T}{\partial y} + \rho C_v u_z \frac{\partial T}{\partial z} &= \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) \\ &+ \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) - T \left.\frac{\partial P}{\partial T}\right|_\rho \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) - \tau_{xx}\frac{\partial u_x}{\partial x} - \tau_{yy}\frac{\partial u_y}{\partial y} - \tau_{zz}\frac{\partial u_z}{\partial z} \\ &- \tau_{xy}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) - \tau_{xz}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) - \tau_{yz}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) + G_e\end{aligned} \tag{F.1-1}$$

where C_v is the heat capacity at constant volume, k the thermal conductivity, and G_e the energy generation rate per unit volume. Equation (F.1-1) can be applied to non-Newtonian fluids if the appropriate constitutive equation relating the viscous stress to the rate of strain is known. For the special case of an incompressible Newtonian fluid with constant thermal conductivity for which the components of the viscous stress tensor are given by equations (D.1-4) through (D.1-9), equation (F.1-1) simplifies to

$$\begin{aligned}\rho C_p \frac{\partial T}{\partial t} + \rho C_p u_x \frac{\partial T}{\partial x} + \rho C_p u_y \frac{\partial T}{\partial y} + \rho C_p u_z \frac{\partial T}{\partial z} &= k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} \\ &+ k\frac{\partial^2 T}{\partial z^2} + 2\mu\left(\frac{\partial u_x}{\partial x}\right)^2 + 2\mu\left(\frac{\partial u_y}{\partial y}\right)^2 + 2\mu\left(\frac{\partial u_z}{\partial z}\right)^2 + \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)^2 \\ &+ \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)^2 + \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)^2 + G_e\end{aligned} \tag{F.1-2}$$

where C_p is the heat capacity at constant pressure.

Scaling Analysis in Modeling Transport and Reaction Processes: A Systematic Approach to Model Building and the Art of Approximation, By William B. Krantz

F.2 CYLINDRICAL COORDINATES

The following form of the thermal energy equation in cylindrical coordinates allows for nonconstant physical properties, energy generation, and conversion of mechanical to internal energy using viscous dissipation, which is expressed in terms of unspecified viscous stress–tensor components τ_{ij}:

$$\begin{aligned}
\rho C_v \frac{\partial T}{\partial t} + \rho C_v u_r \frac{\partial T}{\partial r} + \rho C_v \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + \rho C_v u_z \frac{\partial T}{\partial z} &= \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) \\
&+ \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) - T\left.\frac{\partial P}{\partial T}\right|_\rho \left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right] \\
&- \tau_{rr}\frac{\partial u_r}{\partial r} - \tau_{\theta\theta}\frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) - \tau_{zz}\frac{\partial u_z}{\partial z} - \tau_{r\theta}\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] \\
&- \tau_{rz}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) - \tau_{\theta z}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) + G_e \qquad \text{(F.2-1)}
\end{aligned}$$

where C_v is the heat capacity at constant volume, k the thermal conductivity, and G_e the energy generation rate per unit volume. Equation (F.2-1) can be applied to non-Newtonian fluids if the appropriate constitutive equation relating the viscous stress to the rate of strain is known. For the special case of an incompressible Newtonian fluid with constant thermal conductivity for which the components of the viscous stress tensor are given by equations (D.2-4) through (D.2-9), equation (F.2.1) simplifies to

$$\begin{aligned}
\rho C_p \frac{\partial T}{\partial t} + \rho C_p u_r \frac{\partial T}{\partial r} + \rho C_p \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + \rho C_p u_z \frac{\partial T}{\partial z} &= \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{k}{r^2}\frac{\partial^2 T}{\partial \theta^2} \\
&+ k\frac{\partial^2 T}{\partial z^2} + 2\mu\left(\frac{\partial u_r}{\partial r}\right)^2 + \frac{2\mu}{r^2}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right)^2 + 2\mu\left(\frac{\partial u_z}{\partial z}\right)^2 \qquad \text{(F.2-2)} \\
&+ \mu\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)^2 + \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)^2 + \mu\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right]^2 + G_e
\end{aligned}$$

where C_p is the heat capacity at constant pressure.

F.3 SPHERICAL COORDINATES

The following form of the thermal energy equation in spherical coordinates allows for nonconstant physical properties, energy generation, and conversion of mechanical to internal energy through viscous dissipation, which is expressed in terms of unspecified viscous stress–tensor components τ_{ij}:

$$
\begin{aligned}
\rho C_v \frac{\partial T}{\partial t} &+ \rho C_v u_r \frac{\partial T}{\partial r} + \rho C_v \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + \rho C_v \frac{u_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k \frac{\partial T}{\partial r}\right) \\
&+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) \\
&- T\left.\frac{\partial P}{\partial T}\right|_\rho \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right] \\
&- \tau_{rr}\frac{\partial u_r}{\partial r} - \tau_{\theta\theta}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) - \tau_{\phi\phi}\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta \cot\theta}{r}\right) \\
&- \tau_{r\theta}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right) - \tau_{r\phi}\left(\frac{\partial u_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r}\right) \\
&- \tau_{\theta\phi}\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} - \frac{\cot\theta}{r}u_\phi\right) + G_e \qquad \text{(F.3-1)}
\end{aligned}
$$

where C_v is the heat capacity at constant volume, k the thermal conductivity, and G_e the energy generation rate per unit volume. Equation (F.3-1) can be applied to non-Newtonian fluids if the appropriate constitutive equation relating the viscous stress to the rate of strain is known. For the special case of an incompressible Newtonian fluid with constant thermal conductivity for which the components of the viscous stress tensor are given by equations (D.3-4) through (D.3-9), equation (F.3-1) simplifies to

$$
\begin{aligned}
\rho C_p \frac{\partial T}{\partial t} &+ \rho C_p u_r \frac{\partial T}{\partial r} + \rho C_p \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + \rho C_p \frac{u_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) \\
&+ \frac{k}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{k}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} \\
&+ 2\mu\left(\frac{\partial u_r}{\partial r}\right)^2 + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right)^2 \\
&+ 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r}\right)^2 + \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]^2 \\
&+ \mu\left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right]^2 + \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right]^2 + G_e \\
&\qquad \text{(F.3-2)}
\end{aligned}
$$

where C_p is the heat capacity at constant pressure.