Introductory Concepts

(1) What is pressure?

Pressure, p, is the normal force per unit area exerted on a surface due to the time rate change of momentum of the gas molecules impacting on that surface.

(2) What is temperature?

Temperature, T, is a measure of the average kinetic energy of the particles in the gas. If KE is the mean molecular kinetic energy, then temperature is given by $T = 2\,KE/(3\,k)$, where k is the Boltzmann constant.

(3) What is the relationship between the density and the specific volume of a substance?

Mass per unit volume of a substance is the density, ρ. Volume per unit mass of a substance is the specific volume, v. Thus, the specific volume of a substance is the inverse of its density, that is, $v = 1/\rho$. Therefore, the ideal gas equation of state, $p\,v = R\,T$, shall also be written as $p = \rho\,R\,T$ in which R is the specific gas constant.

(4) Write down the steady flow energy equation and describe the meaning of each term in the equation.

(5) Show that the ideal gas equation can be written as $p = \rho\,R\,T$, and that $\Delta h = c_p\,\Delta T$ for an ideal gas with constant c_p (i.e. the specific heat at constant pressure).

(6) Show that the isentropic flow of an ideal gas with constant specific heats is described by

$$p \propto \rho^{\gamma}, \quad \text{where } \gamma \text{ is the specific heat ratio.}$$

(7) Describe the meaning of (i) steady flow, (ii) adiabatic flow, (iii) inviscid flow, (iv) quasi one-dimensional flow, and (v) compressible flow.

Specific Impulse of a Rocket Engine:

A comparative measure of the efficiency of different rocket engines can be obtained from the specific Impulse I_{sp} defined as the thrust per unit weight flow at sea level. Assuming that the total mass of fuel and oxidizer consumed per unit time, denoted by $\dot{m}$, equals the rate at which product gases of consumption escape at the nozzle exit, we can write

$$I_{sp} = \frac{\dot{m}\,u_e + A_e\,(p_e - p_{amb})}{\dot{m}\,g}$$

where u_e and p_e are the respective velocity and pressure of the exhaust gases at the rocket nozzle exit, A_e is the cross-sectional area of the rocket nozzle exit, p_{amb} is the ambient pressure, and g is the acceleration due to gravity at sea level.

In cases where $p_e \approx p_{amb}$ or in cases where the pressure thrust $A_e\,(p_e - p_{amb})$ is negligible when compared to the momentum thrust $\dot{m}\,u_e$, we can write

$$I_{sp} = \frac{u_e}{g}\,, \qquad \text{where } I_{sp} \text{ takes the unit second.}$$

Mach number, M, at a point in a flow is defined as the ratio between the speed of the flow u at a point and the speed of sound c at the temperature of the flow at the point concerned. If the medium is ideal gas then c is given by $\sqrt{\gamma\,R\,T}$ (the derivation of which is given towards the end of this note set), then

$$M = \frac{u}{\sqrt{\gamma\,R\,T}}$$

where γ is the specific heat ratio, R is the specific gas constant in J/kg K, and T is the temperature of the flow at the point concerned in K.

Important: You now know that the speed of sound in an ideal gas is given by $c = \sqrt{\gamma\,R\,T}$. What are the units of each entity in the above expression?

Introducing M_e, the Mach number of the flow at the rocket nozzle exit, we can write I_{sp} as

$$I_{sp} = \frac{u_e}{g} = \frac{M_e}{g}\,\sqrt{\gamma\,R\,T_e}$$

where T_e is the temperature of the exhaust gases at the rocket nozzle exit.

If the flow through the nozzle is assumed to be a steady, adiabatic, flow of an ideal gas with constant value of c_p (i.e. the specific heat at constant pressure), show that the energy balance yields

$$\frac{T_o}{T_e} = 1 + \frac{\gamma - 1}{2}\,M_e^2 \tag{1}$$

where T_o is the stagnation temperature in the combustion chamber in which the gases are assumed to be at rest or move with negligibly small velocity.

Using the relationship given by equation (1), we can write I_{sp} as

$$I_{sp} = \frac{u_e}{g} = \frac{1}{g}\,\sqrt{\gamma\,R\,T_o\,\left(\frac{M_e^2}{1 + \frac{\gamma-1}{2}\,M_e^2}\right)}\;.$$

The specific gas constant R used in the above expression is determined using

$$\text{specific gas constant } R \text{ in J/kg K} \;=\; \frac{\text{universal gas constant } R \text{ in J/kgmol K}}{Mwt}$$

where Mwt is the average molecular weight of the mixture of exhaust gases flowing through the nozzle.

For a given stagnation temperature T_o and the average molecular weight Mwt, the specific impulse I_{sp} monotonically increases with increasing exit Mach number M_e. Thus, the maximum value of I_{sp}, or of u_e, obtained as M_e reaches infinity is as follows:

$$(I_{sp})_{max} = \frac{(u_e)_{max}}{g} = \frac{1}{g}\sqrt{\gamma\, R\, T_o\,\left(\frac{2}{\gamma-1}\right)}$$

and thus the specific impulse is normalized according to

$$\text{Normalized Specific Impulse} = \frac{I_{sp}}{(I_{sp})_{max}} = \frac{u_e}{(u_e)_{max}} = \sqrt{\frac{\frac{\gamma-1}{2}\,M_e^2}{1+\frac{\gamma-1}{2}\,M_e^2}}\ .$$

The plot below shows how the normalized specific impulse at $\gamma = 1.4$ varies with the exit Mach number M_e.

To evaluate the exit Mach number M_e, let us assume that the steady flow of the ideal gas with constant c_p through the nozzle is not only adiabatic but also reversible, that is, isentropic.

If the flow though the nozzle is assumed to be a steady isentropic flow of an ideal gas with constant c_p then show that $p \propto \rho^{\gamma}$, where ρ is the density, yields

$$\frac{p_o}{p_e} = \left(\frac{T_o}{T_e}\right)^{\frac{\gamma}{\gamma-1}} = \left(1+\frac{\gamma-1}{2}\,M_e^2\right)^{\frac{\gamma}{\gamma-1}} \qquad (2)$$

where p_o is the stagnation pressure in the combustion chamber in which the gases are assumed to be at rest or move with negligibly small velocity.

Rearranging equation (2), we get

$$M_e = \sqrt{\frac{2}{\gamma - 1}\left[\left(\frac{p_o}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}$$

which can be used to evaluate M_e which is to be used in the calculation of I_{sp}. Alternatively, using the relationship given by equation (2), we can rewrite I_{sp} in terms of p_o and p_e as

$$I_{sp} = \frac{u_e}{g} = \frac{1}{g}\sqrt{\frac{2\,\gamma\,R\,T_o}{\gamma - 1}\left[1 - \left(\frac{p_e}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]}\,.$$

Mass flow rate $\dot{m}$:

The steady mass flow rate $\dot{m}$ through the nozzle is the same as the mass flow rate at the exit of the nozzle, and thus

$$\begin{aligned}
\dot{m} &= A_e\,\rho_e\,u_e \\
&= A_e\left(\frac{p_e}{R\,T_e}\right)u_e \qquad \text{(ideal gas behaviour is assumed)} \\
&= A_e\left(\frac{p_e}{R\,T_e}\right)\left(M_e\sqrt{\gamma\,R\,T_e}\right) \qquad \text{(Mach number is introduced)} \\
&= A_e\,p_e\,M_e\sqrt{\frac{\gamma}{R\,T_e}}\,. \qquad (3)
\end{aligned}$$

If the given steady flow of the ideal gas is an adiabatic flow with constant c_p then equation (1) can be used to get

$$\dot{m} = A_e\,p_e\,M_e\sqrt{\frac{\gamma}{R\,T_o}\left(1 + \frac{\gamma - 1}{2}\,M_e^2\right)}\,. \qquad (4)$$

If the given steady flow of the ideal gas is an adiabatic, reversible flow (i.e. isentropic flow) with constant c_p then equation (2) can be used to get the following:

$$\dot{m} = A_e\,M_e\,p_o\sqrt{\frac{\gamma}{R\,T_o}}\left(\frac{1}{1 + \frac{\gamma-1}{2}\,M_e^2}\right)^{\frac{\gamma+1}{2\,(\gamma-1)}} \qquad (5)$$

Problem:

Consider a rocket engine burning hydrogen and oxygen; the combustion chamber pressure and temperature are 25 atm and 3517 K, respectively. The area of the exit is designed such that the exit pressure exactly equals ambient pressure at a standard altitude of 30 km. For the gas mixture, assume $\gamma = 1.22$ and the average molecular weight $Mwt = 16$. At a standard altitude of 30 km, calculate the exit Mach number, specific impulse, and the mass flow rate per unit area of the nozzle exit. Determine also the exit velocity and the thrust per unit area of the nozzle exit. **State clearly all the assumptions made.**

(The ambient pressure at a standard altitude of 30 km is 1.174×10^{-2} atm and 1 atm = 1.013×10^5 N/m^2.)

Governing Equations:

Show that the steady, adiabatic, inviscid, quasi one-dimensional flow is governed by the following equations:

Mass balance: $\dot{m} = A\,\rho\,u =$ constant.
Energy balance: $h + (u^2/2) =$ constant.
Momentum balance: $d\,p + \rho\,u\,du = 0$.

Show that the above set of equations is equivalent to

$$d\,(A\,\rho\,u) = 0, \quad d\,h + u\,du = 0, \quad \text{and} \quad d\,p + \rho\,u\,du = 0\,. \tag{6}$$

Speed of Sound:

Speed of sound is defined as the rate at which infinitesimal disturbances are propagated from their source into an undisturbed medium (such as stagnant air). These disturbances can be thought of as small pressure pulses generated at a point source and propagated in all directions as shown in Figure 1 below. From a molecular viewpoint, of course, it is molecular collisions that must propagate the pressure disturbance and thus the speed of sound is determined by the molecular collisions. To determine the speed of sound c, consider the wave front in Figure 1 propagating into still air that is at pressure p and density ρ. If we chose a very small portion of the curved wave front then we can treat it as planar. Figure 2 shows the movement of the small portion of wave front into still air.

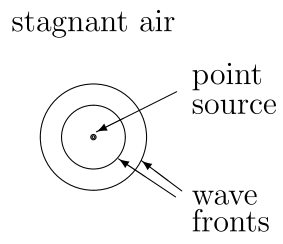

Figure 1

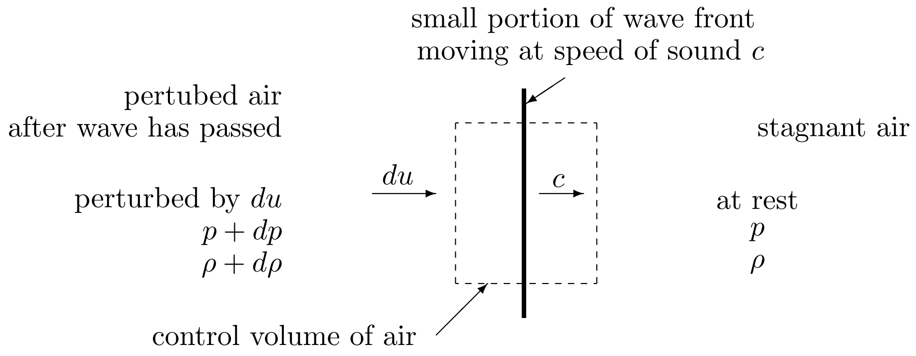

Figure 2

To an observer attached to this small portion of the wave front, the situation appears as shown in Figure 3.

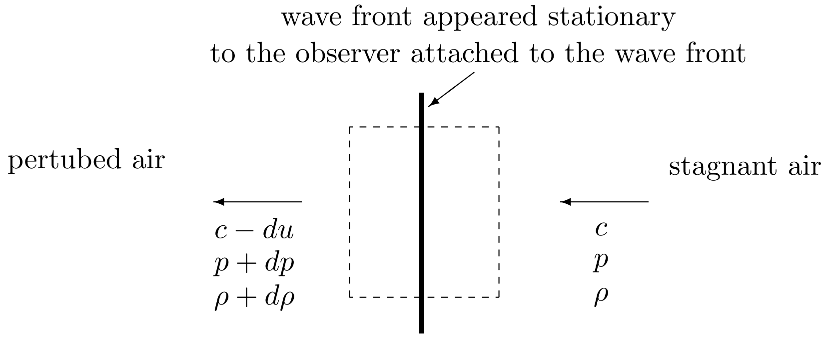

Figure 3

A control volume is also shown attached to the wave both in figure 2 and in Figure 3. The boundaries of the volume are selected so that the flow is normal to faces parallel to the wave and tangent to the other faces. Also, make the assumption that, since the strength of the disturbance in infinitesimal, a fluid particle passing through the wave undergoes a process that is both reversible and adiabatic (i.e. isentropic).

(a) Writing down the mass and momentum balances over the control volume of air, show that $dp = c^2\, d\rho$.

(b) State why the correct form of the above equation is

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s .$$

(c) Prove that for an ideal gas $c = \sqrt{\gamma\, p/\rho} = \sqrt{\gamma\, RT}$, and determine the numerical value of c for air at 300 K.

(d) Prove that $c = \sqrt{p/(\rho \ln \rho)}$ for a highly compressible liquid, the isentropic relationship of which is given by $p \propto \ln \rho$.

Stagnation Properties:

When a fluid is slowed down to rest from some velocity we say that it has attained a *stagnation state*. The properties of the fluid at the stagnation state are known as *stagnation properties* which are denoted with the subscript $_o$ as p_o, T_o, ρ_o, h_o etc. The manner in which the slowing-down process is accomplished obviously influences the stagnation state which will be experienced. The *isentropic stagnation state* is reached by slowing down the fluid isentropically to zero velocity. The *adiabatic stagnation state* is reached by slowing down the fluid adiabatically but *irreversibly* to zero velocity.

We have already seen that the stagnation temperature T_o is related to the temperature T at Mach number M by

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2 ,$$

and that the stagnation pressure p_o is related to the pressure p at Mach number M by

$$\left(\frac{p_o}{p}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{\rho_o}{\rho}\right)^{\gamma-1} = \frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2 .$$

Explain why the temperature equation can be used to describe an adiabatic irreversible process as well as an isentropic process, whereas the pressure equation of can be used to describe an isentropic process only.

Sketches of T/T_o, p/p_o and ρ/ρ_o as a function of the Mach number in the range of $M = 0$ to 5 for air with $\gamma = 1.4$ are given below.

Problems:

(1) Explain in detail how the Mach number of a flow can be obtained with the help of a static probe and a pitot probe in cases of (a) subsonic incompressible flow and (b) subsonic compressible flow.

(2) A high speed subsonic air liner is flying at a pressure altitude of 10 km at which the pressure is 2.65×10^4 N/m^2 according to the standard atmospheric table. A pitot tube on the wing tip measures a pressure of 4.24×10^4 N/m^2. The ambient temperature at the given altitude shall be taken as 230 K. Explain how you would calculate the Mach number at which the air plane is flying (a) ignoring the density changes of air and (b) without ignoring the density changes of air.
(*282 m/s; 258 m/s*)

(3) This problem is taken from page 895 of Çengel & Boles (3^{rd} edition): A subsonic airliner is flying at a 3000-m altitude where the atmospheric conditions are 70.109 kPa and 268.65 K. A pitot tube measures the difference between the static and stagnation pressures to be 14 kPa. Calculate the speed of the airliner and the flight Mach number. (*169.9 m/s; 0.52*)

(4) This problem is taken from page 887 of Çengel & Boles (3^{rd} edition): Air enters a compressor with a stagnation pressure of 100 kPa and a stagnation temperature of 27^{o}C, and it is compressed to a stagnation pressure of 900 kPa. Assuming the compression process to be isentropic, show that the power input to the compressor for a mass flow rate of 0.02 kg/s is 5.27 kW.

(5) This problem is taken from page 847 of Çengel & Boles (3^{rd} edition): An aircraft flying at a cruising speed of 250 m/s at an altitude of 5000 m where the atmospheric pressure is 54.05 kPa and ambient air temperature is 255.7 K. The ambient air is first decelerated in a diffuser before it enters the compressor. Assuming both the diffuser and the compressor to be isentropic, determine (a) the stagnation pressure at the compressor inlet and (b) the required compressor work per unit mass if the stagnation pressure ratio of the compressor is 8.
(*80.79 kPa; 233.9 kJ/kg*)

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