FLUID MECHANICS

TUTORIAL 9

COMPRESSIBLE FLOW

On completion of this tutorial you should be able to

- define entropy
- derive expressions for entropy changes in fluids
- derive Bernoulli's equation for gas
- derive equations for compressible ISENTROPIC flow
- derive equations for compressible ISOTHERMAL flow
- solve problems involving compressible flow
- derive equations for shock waves
- solve problems involving shock waves

Let's start by revising entropy.

1. ENTROPY

1.1 DEFINITION

You should already be familiar with the theory of work laws in closed systems. You should know that the area under a pressure-volume diagram for a reversible expansion or compression gives the work done during the process.

In thermodynamics there are two forms of energy transfer, work (W) and heat (Q). By analogy to work, there should be a property which if plotted against temperature, then the area under the graph would give the heat transfer. This property is entropy and it is given the symbol S. Consider a p-V and T-s graph for a reversible expansion.

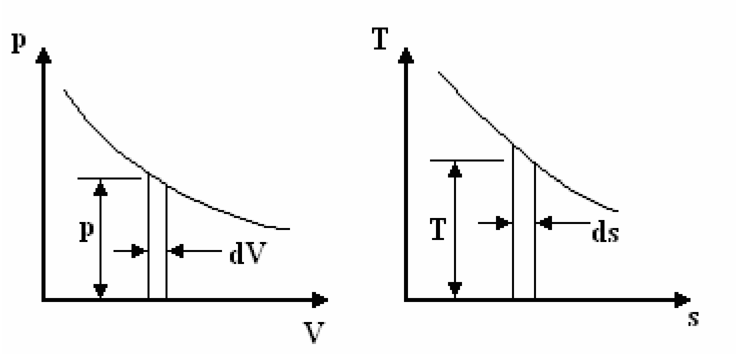

Figure 1

From the p-V graph we have $W = \int pdV$

From the T-S graph we have $Q = \int TdS$

This is the way entropy was developed for thermodynamics and from the above we get the definition

$$dS = dQ/T$$

The units of entropy are hence J/K.

Specific entropy has a symbol s and the units are J/kg K

It should be pointed out that there are other definitions of entropy but this one is the most meaningful for thermodynamics. A suitable integration will enable you to solve the entropy change for a fluid process.

2. ISENTROPIC PROCESSES

The word *Isentropic* means constant entropy and this is a very important thermodynamic process. It occurs in particular when a process is reversible and adiabatic. This means that there is no heat transfer to or from the fluid and no internal heat generation due to friction. In such a process it follows that if dQ is zero then dS must be zero. Since there is no area under the T-S graph, then the graph must be a vertical line as shown.

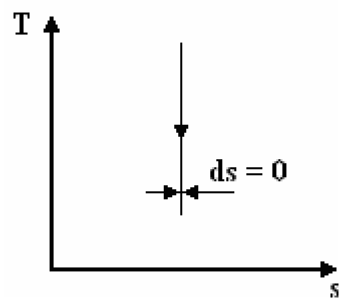

Figure 2

There are other cases where the entropy is constant. For example, if there is friction in the process generating heat but this is lost through cooling, then the nett result is zero heat transfer and constant entropy. You do not need to be concerned about this at this stage.

Entropy is used in the solution of gas and vapour problems. We should now look at practical applications of this property and study the entropy changes which occur in closed and steady flow systems for perfect gases and vapours. These derivations should be learned for the examination.

3. ENTROPY CHANGES FOR A PERFECT GAS IN A CLOSED SYSTEMS

Consider a closed system expansion of a fluid against a piston with heat and work transfer taking place.

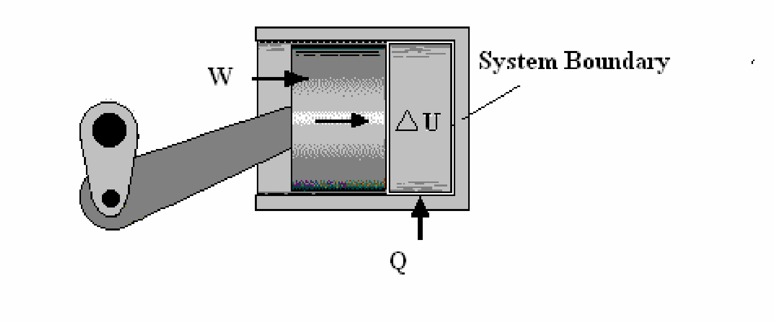

Figure 3

Applying the non-flow energy equation we have

$$Q + W = \Delta U$$

Differentiating we have $dQ + dW = dU$

Since dQ = TdS and dW = -pdV then $TdS - pdV = dU$

$$TdS = dU + pdV$$

This expression is the starting point for all derivations of entropy changes for any fluid (gas or vapour) in closed systems. It is normal to use specific properties so the equation becomes

$$Tds = du + pdv$$

but from the gas law pv = RT we may substitute for p and the equation becomes

$$Tds = du + RTdv/v$$

rearranging and substituting $du = c_V\, dT$ we have

$$ds = c_V\, dT/T + Rdv/v \quad \text{...............(1)}$$

s is specific entropy

v is specific volume.

u is specific internal energy and later on is also used for velocity.

3.1 ISOTHERMAL PROCESS

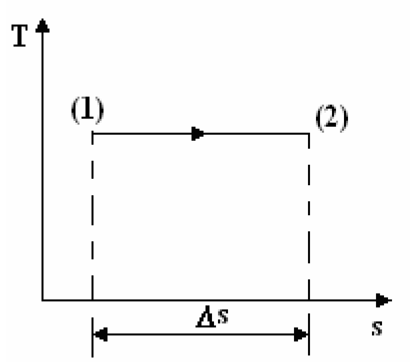

Figure 4

In this case temperature is constant. Starting with equation (1)

$$ds = c_v\, dT/T + R dv/v.$$

since dT = 0 then

$$s_2 - s_1 = \Delta s = R \ln(v_2/v_1)$$

A quicker alternative derivation for those familiar with the work laws is:

$Q + W = \Delta U$ but $\Delta U = 0$ then $Q = -W$ and $W = -mRT \ln\frac{V_2}{V_1}$

$Q = \int T ds = T\Delta S$ but T is constant.

$$\Delta S = \frac{Q}{T} = -\frac{W}{T} = mR\ln\frac{V_2}{V_1}$$

$$\Delta S = mR\ln\frac{V_2}{V_1}$$

$\Delta s = R\ln\frac{v_2}{v_1}$ and since $\frac{v_2}{v_1} = \frac{p_1}{p_2}$

$$\Delta s = R\ln\frac{p_1}{p_2}$$

3.2 CONSTANT VOLUME PROCESS

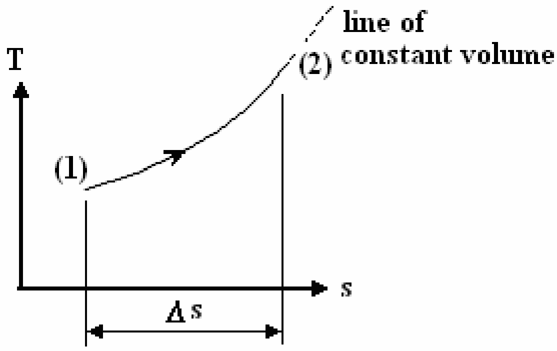

Figure 5

Starting again with equation (1) we have $ds = c_v dT/T + R dv/v$

In this case dv=0 so $ds = c_v dT/T$

Integrating between limits (1) and (2) $\Delta s = c_v \ln(T_2/T_1)$

3.3 CONSTANT PRESSURE PROCESS

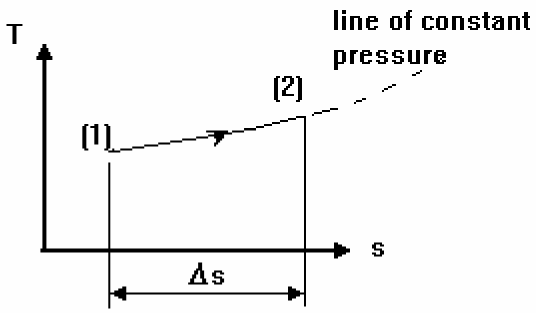

Figure 6

Starting again with equation (1) we have

$ds = C_v \frac{dT}{T} + R\frac{dv}{v}$ In this case we integrate and obtain

$\Delta s = C_v \ln\frac{T_2}{T_1} R\ln\frac{v_2}{v_1}$ For a constant pressure process, v/T = constant

$\frac{v_2}{v_1} = \frac{T_2}{T_1}$ so the expression becomes $\Delta s = C_v \ln\frac{T_2}{T_1} + R\ln\frac{T_2}{T_1} = (C_v + R)\ln\frac{T_2}{T_1}$

It was shown in an earlier tutorial that $R = c_p - c_v$ hence

$$\Delta s = C_p \ln\frac{T_2}{T_1}$$

3.4 POLYTROPIC PROCESS

This is the most difficult of all the derivations here. Since all the forgoing are particular examples of the polytropic process then the resulting formula should apply to them also.

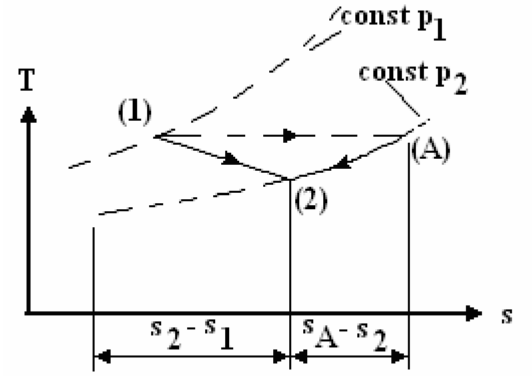

Figure 7

The polytropic expansion is from (1) to (2) on the T-s diagram with different pressures, volumes and temperatures at the two points. The derivation is done in two stages by supposing the change takes place first at constant temperature from (1) to (A) and then at constant pressure from (A) to (2). You could use a constant volume process instead of constant pressure if you wish.

$s_2-s_1 = (s_A-s_1) - (s_A-s_2)$

$s_2-s_1 = (s_A-s_1) + (s_2-s_A)$

For the constant temperature process

$(s_A-s_1) = R \ln(p_1/p_A)$

For the constant pressure process

$(s_2-s_A) = (c_p) \ln(T_2/T_A)$

Hence

$$\Delta s = R\ln\frac{p_1}{p_A} + C_p \ln\frac{T_2}{T_A} + s_2\text{-}s_1$$ Since $p_A = p_2$ and $T_A = T_1$

Then

$$\Delta s = s_2\text{-}s_1 = R\ln\frac{p_1}{p_2} + C_p \ln\frac{T_2}{T_1}$$ Divide through by R

$$\frac{\Delta s}{R} = \ln\frac{p_1}{p_2} + \frac{C_p}{R}\ln\frac{T_2}{T_1}$$

From the relationship between c_p, c_v, R and γ we have $c_p/R = \gamma/(\gamma-1)$

Hence $$\frac{\Delta s}{R} = \ln\frac{p_1}{p_2} + \frac{\gamma}{\gamma-1}\ln\frac{T_2}{T_1} \qquad \frac{\Delta s}{R} = \ln\frac{p_1}{p_2}\left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

This formula is for a polytropic process and should work for isothermal, constant pressure, constant volume and adiabatic processes also. In other words, it must be the derivation for the entropy change of a perfect gas for any closed system process. This derivation is often requested in the exam.

WORKED EXAMPLE No. 1

A perfect gas is expanded from 5 bar to 1 bar by the law $pV^{1.2} = C$. The initial temperature is 200°C. Calculate the change in specific entropy.
R = 287 J/kg K γ =1.4.

SOLUTION

$$T_2 = 473\left(\frac{1}{5}\right)^{1-\frac{1}{1.2}} = 361.7\text{K}$$

$$\frac{\Delta s}{R} = \ln\left(\frac{p_1}{p_2}\right)\left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\Delta s}{R} = (\ln 5)\left(\frac{361.7}{472}\right)^{3.5} = 0.671$$

$$\Delta s = 0.671 \text{ x } 287 = 192.5 \text{ J/kgK}$$

SELF ASSESSMENT EXERCISE No. 1

1. Calculate the specific entropy change when a perfect gas undergoes a reversible isothermal expansion from 500 kPa to 100 kPa. R = 287 J/kg K.
(Answer +461.9 J/kg K).

2. Calculate the total entropy change when 2 kg of perfect gas is compressed reversibly and isothermally from 9 dm^3 to 1 dm^3. R=300 J/kg K. (Answer -1.32 kJ/k)

3. Calculate the change in entropy when 2.5 kg of perfect gas is heated from 20°C to 100°C at constant volume. Take c_V= 780 J/kg K (Answer 470 J/K)

4. Calculate the total entropy change when 5 kg of gas is expanded at constant pressure from 30°C to 200°C. R = 300 J/kg K c_V= 800 J/kg K (Answer 2.45 kJ/K)

5. Derive the formula for the specific change in entropy during a polytropic process using a constant volume process from (A) to (2).

6. A perfect gas is expanded from 5 bar to 1 bar by the law $pV^{1.6} = C$. The initial temperature is 200°C. Calculate the change in specific entropy.
R = 287 J/kg K γ =1.4.   (Answer -144 J/kg K)

7. A perfect gas is expanded reversibly and adiabatically from 5 bar to 1 bar by the law $pV^{\gamma} = C$. The initial temperature is 200°C. Calculate the change in specific entropy using the formula for a polytropic process. R = 287 J/kg K γ =1.4. (The answer should be zero since the process is constant entropy).

Let's go on to apply the knowledge of entropy to the flow of compressible fluids starting with isentropic flow.

4. ISENTROPIC FLOW

Isentropic means constant entropy. In this case we will consider the flow to be ADIABATIC also, that is, with no heat transfer.

Consider gas flowing in a duct which varies in size. The pressure and temperature of the gas may change.

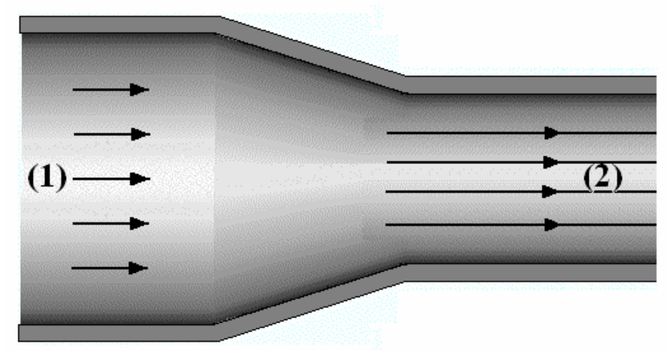

Figure 8

Applying the steady flow energy equation between (1) and (2) we have :

$$\Phi - P = \Delta U + \Delta F.E. + \Delta K.E. + \Delta P.E.$$

For Adiabatic Flow, $\Phi = 0$ and if no work is done then $P = 0$

$$\Delta U + \Delta F.E. = \Delta H$$

hence :

$$0 = \Delta H + \Delta K.E. + \Delta P.E.$$

In specific energy terms this becomes :

$$0 = \Delta h + \Delta k.e. + \Delta p.e.$$

rewriting we get:

$$h_1 + u_1^2/2 + g z_1 = h_2 + u_2^2/2 + g z_2$$

For a gas, $h = C_p T$ so we get Bernoulli's equation for gas which is :

$$C_pT_1 + u_1^2/2 + g z_1 = C_pT_2 + u_2^2/2 + g z_2$$

Note that T is absolute temperature in Kelvins $T = {}^oC + 273$

4.1 STAGNATION CONDITIONS

If a stream of gas is brought to rest, it is said to STAGNATE. This occurs on leading edges of any obstacle placed in the flow and in instruments such as a Pitot Tube. Consider such a case for horizontal flow in which P.E. may be neglected.

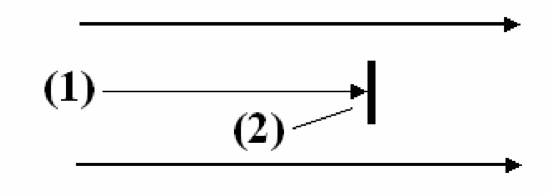

Figure 9

$u_2 = 0$ and $z_1 = z_2$ so $C_pT_1 + u_1^2/2 = C_pT_2 + 0$

$T_2 = u_1^2/2C_p + T_1$

T_2 is the stagnation temperature for this case.

$$\text{Let } T_2 - T_1 = \Delta T = u_1^2/2C_p$$

$$\Delta T = u_1^2/2C_p$$

Now $C_p - C_v = R$ and $C_p / C_v = \gamma$ γ is the adiabatic index .

hence $C_p = R / (\gamma - 1)$ and so :

$$\Delta T = u_1^2 (\gamma - 1) / (2\gamma R)$$

It can be shown elsewhere that the speed of sound a is given by :

$$a^2 = \gamma RT$$

hence at point 1:

$$\Delta T / T_1 = u_1^2 (\gamma - 1) / (2\gamma RT_1) = u_1^2 (\gamma - 1) /2a_1^2$$

The ratio u/a is the Mach Number M_a so this may be written as :

$$\Delta T / T_1 = M_a^2 (\gamma - 1) /2$$

If M_a is less than 0.2 then M_a^2 is less than 0.04 and so $\Delta T/T_1$ is less than 0.008. It follows that for low velocities, the rise in temperature is negligible under stagnation conditions.

The equation may be written as :

$$\frac{T_2 - T_1}{T_1} = \frac{M_a^2(\gamma - 1)}{2}$$

$$\frac{T_2}{T_1} = \left\{\frac{M_a^2(\gamma - 1)}{2}\right\} + 1$$

Since pV/T = constant and pV^{γ} = constant then :

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

Hence :

$$\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \frac{M_a^2(\gamma - 1)}{2} + 1$$

$$\left(\frac{p_2}{p_1}\right) = \left[\frac{M_a^2(\gamma - 1)}{2} + 1\right]^{\frac{\gamma}{\gamma-1}}$$

p_2 is the stagnation pressure. If we now expand the equation using the binomial theorem we get :

$$\frac{p_2}{p_1} = 1 + \frac{\gamma Ma^2}{2}\left\{1 + \frac{Ma^2}{4} + \frac{Ma^4}{8} + \ldots\ldots\ldots\right\}$$

If M_a is less than 0.4 then : $\frac{p_2}{p_1} = 1 + \frac{\gamma Ma^2}{2}$

Now compare the equations for gas and liquids :

LIQUIDS $u = (2\Delta p/\rho)^{0.5}$

GAS $\frac{p_2}{p_1} = 1 + \frac{\gamma Ma^2}{2}$

Put $p_2 = p_1 + \Delta p$ so : $\Delta p = \frac{\gamma Ma^2}{2} p_1 = \frac{\gamma v_1^2 p_1}{2\gamma RT} = \frac{\rho_1 u_1^2}{2}$

where $\rho_1 = p_1/RT$ and $M_a^2 = u_1^2/(\gamma RT)$

hence $u = (2\Delta p/\rho_1)^{0.5}$ which is the same as for liquids.

SELF ASSESSMENT EXERCISE No. 2

Take $\gamma = 1.4$ and R = 283 J/kg K in all the following questions.

1. An aeroplane flies at Mach 0.8 in air at 15^o C and 100 kPa pressure. Calculate the stagnation pressure and temperature. (Answers 324.9 K and 152.4 kPa)

2. Repeat problem 1 if the aeroplane flies at Mach 2. (Answers 518.4 K and 782.4 kPa)

3. The pressure on the leading edges of an aircraft is 4.52 kPa more than the surrounding atmosphere. The aeroplane flies at an altitude of 5 000 metres. Calculate the speed of the aeroplane.(Answer 109.186 m/s)

   Note from fluids tables, find that a = 320.5 m/s p_1 = 54.05 kPa γ = 1.4

4. An air compressor delivers air with a stagnation temperature 5 K above the ambient temperature. Determine the velocity of the air. (Answer 100.2 m/s)

Let's now extend the work to pitot tubes.

5. PITOT STATIC TUBE

A Pitot Static Tube is used to measure the velocity of a fluid. It is pointed into the stream and the differential pressure obtained gives the stagnation pressure.

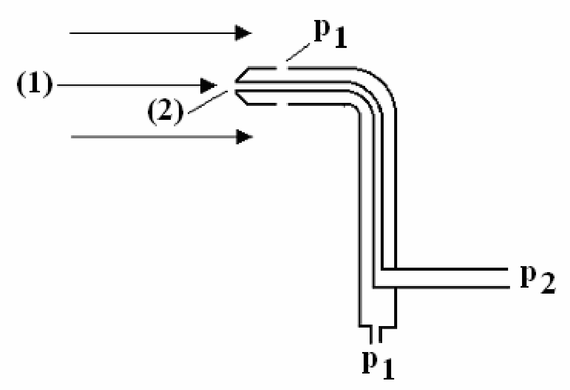

Figure 10

$p_2 = p_1 + \Delta p$

Using the formula in the last section, the velocity v may be found.

WORKED EXAMPLE No.2

A pitot tube is pointed into an air stream which has a pressure of 105 kPa. The differential pressure is 20 kPa and the air temperature is 20°C. Calculate the air speed.

SOLUTION

$p_2 = p_1 + \Delta p = 105 + 20 = 125$ kPa

$$\frac{p_2}{p_1} = \left[\left\{\frac{Ma^2(\gamma - 1)}{2}\right\} + 1\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{125}{105} = \left[\left\{\frac{Ma^2(\gamma - 1)}{2}\right\} + 1\right] \text{hence Ma} = 0.634$$

$a = (\gamma RT)^{0.5} = (1.4 \times 287 \times 293)^{0.5} = 343$ m/s

$M_a = u/a$ hence $u = 217.7$ m/s

Let's further extend the work now to venturi meters and nozzles.

6. VENTURI METERS AND NOZZLES

Consider the diagrams below and apply Isentropic theory between the inlet and the throat.

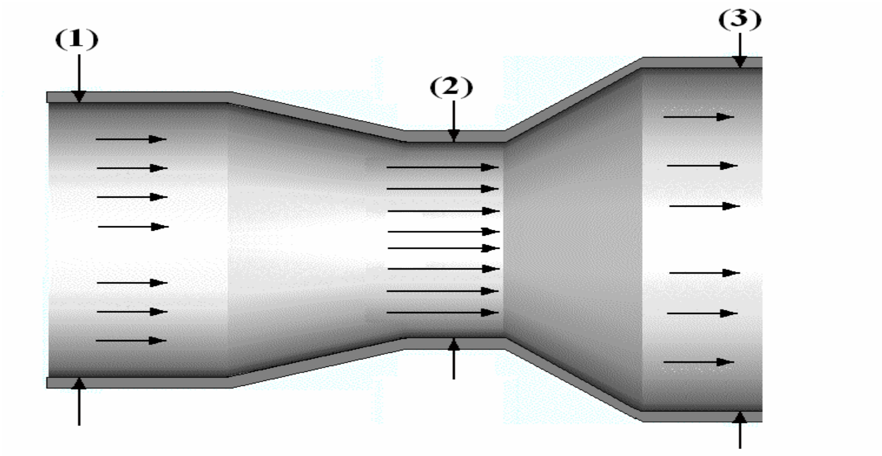

Figure 11

$u_2^2 - u_1^2 = h_1 - h_2$

If the Kinetic energy at inlet is ignored this gives us $u_2^2 = h_1 - h_2$

For a gas h = C_pT so: $u_2^2 = C_P[T_1 - T_2]$

Using $C_p = \gamma R/(\gamma-1)$ we get $u_2^2 = \frac{2\gamma R}{\gamma - 1}[T_1 - T_2]$

$RT = pV/m = p/\rho$ so

$$u_2^2 = \frac{2\gamma}{\gamma - 1}\left[\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right]$$

$$p_1V_1^\gamma = p_2V_2^\gamma \text{ so it follows that } \frac{p_1}{\rho_1^\gamma} = \frac{p_2}{\rho_2^\gamma}$$

$$u_2^2 = \frac{2\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1}\right)\left[1 - \frac{p_2\rho_1}{p_1\rho_2}\right]$$

$$u_2^2 = \frac{2\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1}\right)\left[1 - \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right]$$

The mass flow rate m $=\rho_2 A_2 u_2 C_d$ where C_d is the coefficient of discharge which for a well designed nozzle or Venturi is the same as the coefficient of velocity since there is no contraction and only friction reduces the velocity.

$$\rho_2 = \rho_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}} \quad \text{hence} \quad m = C_d A_2\sqrt{\left[\frac{2\gamma}{\gamma - 1}\right]\left\{[p_1\rho_1]\left[\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}} - \left(\frac{p_2}{p_1}\right)^{1+\frac{1}{\gamma}}\right]\right\}}$$

If a graph of mass flow rate is plotted against pressure ratio (p_2/ p_1) we get:

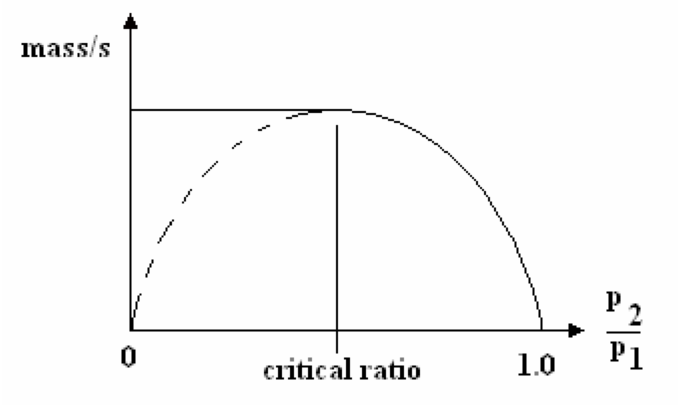

Figure 12

Apparently the mass flow rate starts from zero and reached a maximum and then declined to zero. The left half of the graph is not possible as this contravenes the 2nd law and in reality the mass flow rate stays constant over this half.

What this means is that if you started with a pressure ratio of 1, no flow would occur. If you gradually lowered the pressure p_2 , the flow rate would increase up to a maximum and not beyond. The pressure ratio at which this occurs is the CRITICAL RATIO and the nozzle or Venturi is said to be choked when passing maximum flow rate. Let $\frac{p_2}{p_1} = r$

For maximum flow rate, $\frac{dm}{dr} = 0$

The student should differentiate the mass formula above and show that at the maximum condition the critical pressure ratio is :

$$r = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}}$$

6.1 MAXIMUM VELOCITY

If the formula for the critical pressure ratio is substituted into the formula for velocity, then the velocity at the throat of a choked nozzle/Venturi is :

$$u_2^2 = \left\{\frac{\gamma p_2}{\rho_2}\right\} = \gamma RT = a^2$$

Hence the maximum velocity obtainable at the throat is the local speed of sound.

6.2 CORRECTION FOR INLET VELOCITY

In the preceding derivations, the inlet velocity was assumed negligible. This is not always the case and especially in Venturi Meters, the inlet and throat diameters are not very different and the inlet velocity should not be neglected. The student should go through the derivation again from the beginning but this time keep v_1 in the formula and show that the mass flow rate is

$$m = \frac{C_d A_2 \sqrt{\left[p_1 \rho_1 \frac{2\gamma}{\gamma - 1}\right]\left[\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}} - \left(\frac{p_2}{p_1}\right)^{1+\frac{1}{\gamma}}\right]}}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2 \left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}}}}$$

The critical pressure ratio can be shown to be the same as before.

6.3 MORE ON ISENTROPIC FLOW

When flow is isentropic it can be shown that all the stagnation properties are constant. Consider the conservation of energy for a horizontal duct :

$h + u^2/2 = \text{constant}$ $\quad\quad$ h = specific enthalpy

If the fluid is brought to rest the total energy must stay the same so the stagnation enthalpy h_o is given by :

$h_o = h + u^2/2$ and will have the same value at any point in the duct.

since $h_o = C_p T_o$ then To (the stagnation temperature) must be the same at all points. It follows that the stagnation pressure p_o is the same at all points also. This knowledge is very useful in solving questions.

6.4 ISENTROPIC EFFICIENCY (NOZZLE EFFICIENCY)

If there is friction present but the flow remains adiabatic, then the entropy is not constant and the nozzle efficiency is defined as :

$$\eta = \text{actual enthalpy drop/ideal enthalpy drop}$$

For a gas this becomes : $(T_1 - T_2)/(T_1 - T_2')$

T_2' is the ideal temperature following expansion. Now apply the conservation of energy between the two points for isentropic and non isentropic flow :

$$C_p T_1 + u_1^2/2 = C_p T_2 + u_2^2/2 \quad \text{....... for isentropic flow}$$

$$C_p T_1 + u_1^2/2 = C_p T_{2'} + u_{2'}^2/2 \quad \text{.........for non isentropic}$$

Hence

$$\eta = (T_1 - T_2)/(T_1 - T_2') = (u_2^2 - u_1^2)/(u_{2'}^2 - u_1^2)$$

If v_1 is zero (for example Rockets) then this becomes :

$$\eta = u_2^2 / u_{2'}^2$$

SELF ASSESSMENT EXERCISE No. 3

1. A Venturi Meter must pass 300g/s of air. The inlet pressure is 2 bar and the inlet temperature is 120°C. Ignoring the inlet velocity, determine the throat area. Take C_d as 0.97. Take γ =1.4 and R = 287 J/kg K (assume choked flow)
(Answer 0.000758 m^2)

2. Repeat problem 1 given that the inlet is 60 mm diameter and the inlet velocity must not be neglected. (Answer 0.000747 m^2)

3. A nozzle must pass 0.5 kg/s of steam with inlet conditions of 10 bar and 400°C. Calculate the throat diameter that causes choking at this condition. The density of the steam at inlet is 3.263 kg/m^3. Take γ for steam as 1.3 and C_d as 0.98.
(Answer 23.2 mm)

4. A Venturi Meter has a throat area of 500 mm^2. Steam flows through it, and the inlet pressure is 7 bar and the throat pressure is 5 bar. The inlet temperature is 400°C. Calculate the flow rate. The density of the steam at inlet is 2.274 kg/m^3.

   Take γ= 1.3. R = 462 J/kg K. Cd=0.97. (Answer 383 g/s)

5. A pitot tube is pointed into an air stream which has an ambient pressure of 100 kPa and temperature of 20°C. The pressure rise measured is 23 kPa. Calculate the air velocity. Take γ = 1.4 and R = 287 J/kg K. (Answer 189.4 m/s)

6. A fast moving stream of gas has a temperature of 25°C. A thermometer is placed into it in front of a small barrier to record the stagnation temperature. The stagnation temperature is 28°C. Calculate the velocity of the gas. Take γ= 1.5 and R = 300 J/kg K. (Answer 73.5 m/s)

Let's do some further study of nozzles of venturi shapes now.

7. CONVERGENT - DIVERGENT NOZZLES

A nozzle fitted with a divergent section is in effect a Venturi shape. The divergent section is known as a diffuser.

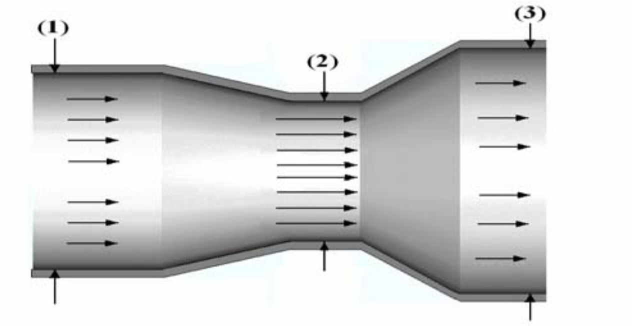

Figure 13

If p_1 is constant and p_3 is reduced in stages, at some point p_2 will reach the critical value which causes the nozzle to choke. At this point the velocity in the throat is sonic.

If p_3 is further reduced, p_2 will remain at the choked value but there will be a further pressure drop from the throat to the outlet. The pressure drop will cause the volume of the gas to expand. The increase in area will tend to slow down the velocity but the decrease in volume will tend to increase the velocity. If the nozzle is so designed, the velocity may increase and become supersonic at exit.

In rocket and jet designs, the diffuser is important to make the exit velocity supersonic and so increase the thrust of the engine.

7.1 NOZZLE AREAS

When the nozzle is choked, the velocity at the throat is the sonic velocity and the Mach number is 1. If the Mach number at exit is M_e then the ratio of the throat and exit area may be found easily as follows.

$u_t = (\gamma RT_t)^{0.5}$ $u_e = M_e(\gamma RT_e)^{0.5}$ mass/s = $\rho_t A_t v_t = \rho_e A_e v_e$.

$\frac{A_t}{A_e} = \frac{\rho_e u_e}{\rho_t u_t}$ but earlier it was shown that $\frac{\rho_e}{\rho_t} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}}$

$\frac{A_t}{A_e} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}} \frac{M_e(\gamma RT_e)^{0.5}}{(\gamma RT_t)^{0.5}}$ It was also shown earlier that $\frac{T_e}{T_t} = \left(\frac{p_e}{p_t}\right)^{1-\frac{1}{\gamma}}$

$$\frac{A_t}{A_e} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}} M_e \left\{\left(\frac{p_e}{p_t}\right)^{1-\frac{1}{\gamma}}\right\}^{0.5}$$

$$\frac{A_t}{A_e} = M_e \left(\frac{p_e}{p_t}\right)^{\frac{1+\gamma}{2\gamma}}$$

There is much more which can be said about nozzle design for gas and steam with implications to turbine designs. This should be studied in advanced text books.

WORKED EXAMPLE No.3

Solve the exit velocity for the nozzle shown assuming isentropic flow:

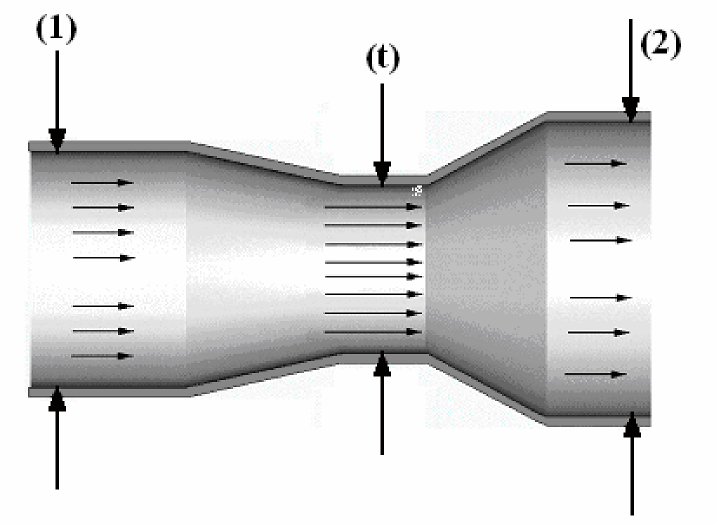

Figure 14

$T_1 = 350$ K $P_1 = 1$ MPa $p_2 = 100$ kPa

The nozzle is fully expanded (choked). Hence $M_t = 1$ (the Mach No.)
The adiabatic index $\gamma = 1.4$

SOLUTION

The critical pressure $p_t = p_1 \{2/(\gamma - 1)\}^{\gamma/(\gamma-1)} = 0.528$ MPa

$T_t/T_1 = (p_t/p_1)^{(\gamma-1)/\gamma}$ hence $T_t = 291.7$ K
$T_o/T_t = \{1 + M^2(\gamma - 1)/2\}$ hence $T_o = 350$ K

It makes sense that the initial pressure and temperature are the stagnation values since the initial velocity is zero.

$T_2 = T_t (p_2/p_t)^{(\gamma-1)/\gamma} = 181.3$ K $a_2 = (\gamma R T_2)^{0.5} = 270$ m/s

$p_o/p_2 = \{1 + M_2^2(\gamma - 1)/2\}^{\gamma/(\gamma-1)}$

Hence $M_2 = 2.157$ and $u_2 = 2.157 \times 270 = 582.4$ m/s

SELF ASSESSMENT EXERCISE No. 4

1. Air discharges from a pipe into the atmosphere through an orifice. The stagnation pressure and temperature immediately upstream of the orifice is 10 bar and 287 K at all times.

   Determine the diameter of the orifice which regulates the flow rate to 0.03 kg/s. (Answer 4 mm)

   Determine the diameter of the orifice which regulates the flow rate to 0.0675 kg/s. (Answer 6 mm)

   Atmospheric pressure is 1 bar, the flow is isentropic and the air should be treated as a perfect gas. The following formulae are given to you.

   $T_o = T\{1 + M^2(\gamma-1)/2\}$ $\quad\quad$ $p_1/p_2 = (T_1/T_2)^{\gamma/(\gamma-1)}$

   The relationship between areas for the flow of air through a convergent- divergent nozzle is given by

   $A/A^* = (1/M)\{(M^2 + 5)/6\}^3$

   where A and A* are cross sectional areas at which the Mach Numbers are M and 1.0 respectively.

   Determine the ratio of exit to throat areas of the nozzle when the Mach number is 2.44 at exit. (Answer 2.49/1)

   Confirm that an exit Mach number of 0.24 also gives the same area ratio.

2. Air discharges from a vessel in which the stagnation temperature and pressure are 350 K and 1.3 bar into the atmosphere through a convergent-divergent nozzle. The throat area is 1×10^{-3} m^2. The exit area is 1.2×10^{-3} m^2. Assuming isentropic flow and no friction and starting with the equations

   $$a = (\gamma RT)^{1/2}$$
   $$C_p T_o = C_p T + v_2/2$$
   $$p\rho^{-\gamma} = \text{constant}$$

   Determine the mass flow rate through the nozzle , the pressure at the throat and the exit velocity. (Answers 0.28 kg/s, 0.686 bar, 215.4 m/s)

3. Show that the velocity of sound in a perfect gas is given by $a = (\gamma RT)^{½}$

Show that the relationship between stagnation pressure, pressure and Mach number for the isentropic flow of a perfect gas is

$p_o/p = \{1 + (\gamma\text{-}1)M^2/2\}^{\gamma/(\gamma-1)}$

It may be assumed that $ds = C_p\ d(l_n v) + C_v\ d(l_n p)$

where v is the specific volume.

A convergent - divergent nozzle is to be designed to produce a Mach number of 3 when the absolute pressure is 1 bar. Calculate the required supply pressure and the ratio of the throat and exit areas.

(Answers 36.73 bar 0.236/1)

Let's now examine the flow of gases in long pipes and ducts in which the temperature stays constant.

8. ISOTHERMAL FLOW

Isothermal flow normally occurs in long pipes in which the temperature of the gas has time to normalise with the surroundings. Consider a section of such a pipe :

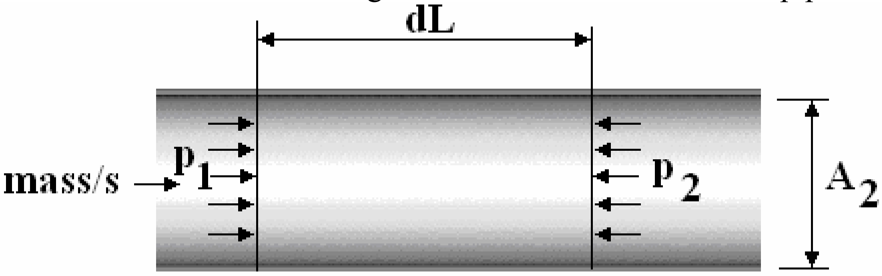

Figure 15

The friction coefficient is C_f as defined by D'Arcy's formula.

$p_2 - p_1 = dp$ Pressure force = $A(p_1 - p_2) = -A\,dp$

Resisting shear force = $\tau_w \pi D dL$ where τ_w is the wall shear stress and D is the diameter.

Since the pressure drops along the length, the volume expands and so the velocity increases since the area is constant du = increase in velocity.

If the velocity increases, then the mass must be accelerated. the inertia force required to this is equal to the change in momentum per second $m(u_2 - u_1)$

Balancing forces produces this equation :

$$-A\,dp - \pi\,D\,dL\,\tau_w = m(u_2 - u_1) = \rho_2 A u_2 \quad \text{divide by A and put } A = \frac{\pi D^2}{4}$$

$$-dp = \frac{4\pi D\,dL\,\tau_w}{\pi D^2} + \rho_2 u_2 (u_2 - u_1) \quad \text{let the change in velocity be very small so that } (u_2 - u_1) = du$$

$$-dp = \frac{4\,dL\,\tau_w}{D} + \rho_2 u_2 (du)$$

The friction coefficient C_f is defined as :

C_f = Wall shear Stress/dynamic pressure = $2\,\tau_w / \rho\,u^2$

$$-dp = \frac{4\,dL\rho_2\,u_2^2 C_f}{2D} + \rho_2 u_2 (du) \quad \text{divide through by } \rho_2\,u_2^2$$

$$\frac{-dp}{\rho_2\,u_2^2} = \frac{2\,dLC_f}{D} + \frac{du}{u}$$

We may drop the suffix so that for any given point in the pipe

$$\frac{-dp}{\rho\,u^2} = \frac{2\,dLC_f}{D} + \frac{du}{u}$$

$-dp / (\rho u^2) = (2\,C_f\,dL / D) + (du / u)$

Usually the change in velocity is negligible and dv is approximately zero. This reduces the equation to :

$-dp / (\rho u^2) = (2 C_f dL / D)$

Hence $-dp / dL = (\rho u^2)(2 C_f / D)$

Since V = mRT/p = Av then v = mRT/pA = 4mRT/pπD² where m is the mass flow rate

and so $u^2 = 16m^2R^2T^2/p^2\pi^2D^4$

and $\rho u^2 = 16 \rho m^2R^2T^2/p^2\pi^2D^4$

but $\rho = p/RT$ so $\rho u^2 = 16 \rho m^2 RT/p\pi^2 D^4$

so $-dp/dL = (32 C_f m^2 RT)/(p\pi^2 D^5)$

$-pdp = (32 C_f m^2 RT)(dL)/\pi^2 D^5)$

Integrating with corresponding limits of L=0 when $p = p_1$ and L=L when $p = p_2$

Then $(1 - p_2^2/p_1^2) = (64 m^2 RT C_f L)/(\pi^2 D^5 p_1^2)$

SELF ASSESSMENT EXERCISE No. 5

1. An air storage vessel contains air at 6.5 bar and 15°C. Air is supplied from the vessel to a machine through a pipe 90 m long and 50 mm diameter. The flow rate is 2.25 m^3/min at the pipe inlet. The friction coefficient C_f is 0.005. Neglecting kinetic energy, calculate the pressure at the machine assuming isothermal flow. (Answer 5.98 bar)

8.1 FRICTION COEFFICIENT

The friction coefficient C_f has been comprehensively explained in other tutorials for non - compressible flow.

For smooth bore pipes the following is found to be accurate.

BLAZIUS found $C_f = 0.079 Re^{-0.25}$

LEE found that $C_f = 0.0018 + Re^{-0.35}$

Otherwise use the Moody chart to find f in which case you need to remember that

$Re = \rho u D/\mu$ but since $u = V/A$ and $\rho = m/V$ then $Re = 4m/\rho u D$

ASSIGNMENT 6

1. A natural gas pipeline is 1000 m long and 100 mm bore diameter. It carries 0.7 kg/s of gas at a constant temperature of 0^o C. The viscosity is 10.3×10^{-6} N s/m^2 and the gas constant R = 519.6 J/kg K. The outlet pressure is 105 kPa. Calculate the inlet pressure. using the Blazius formula to find f. (Answer 357 kPa.)

2. A pipeline is 20 km long and 500 mm bore diameter. 3 kg/s of natural gas must be pumped through it at a constant temperature of 20^oC. The outlet pressure is 200 kPa. Calculate the inlet pressure using the same gas constants as Q.1.
(Answer 235 kPa)

3. Air flows at a mass flow rate of 9.0 kg/s isothermally at 300 K through a straight rough duct of constant cross sectional area of 1.5×10^{-3} m^2. At end A the pressure is 6.5 bar and at end B it is 8.5 bar. Determine

    a. the velocities at each end. (Answers 794.8m/s and 607.7 m/s)

    b. the force on the duct. (Answer 1 380 N)

    c. the rate of heat transfer through the walls. (Answer 1.18 MJ)

    d. the entropy change due to heat transfer. (Answer 3.935 KJ/k)

    e. the total entropy change. (Answer 0.693 kJ/K)

It may be assumed that $ds = C_p\, dT/T + R\, dp/p$

4. A gas flows along a pipe of diameter D at a rate of m kg/s. Show that the pressure gradient is
$-dp/dL = (32\ fm^2RT)/(p\pi^2D^5)$

Methane gas is passed through a pipe 500 mm diameter and 40 km long at 13 kg/s. The supply pressure is 11 bar. The flow is isothermal at 15°C. Given that the molecular mass is 16 kg/kmol and the friction coefficient C_f is 0.005 determine

a. the exit pressure. (Answer 3.99 bar)

b. the inlet and exit velocities. (Answers 9.014 m/s and 24.85 m/s)

c. the rate of heat transfer to the gas. (Answer 3.48 kW)

d. the entropy change resulting from the heat transfer. (Answer 12.09 kJ/K)

e. the total entropy change calculated from the formula

$ds = C_p \ln(T_2/T_1) - R \ln(p_2/p_1)$ (Answer 1.054 kJ/K)

Let's now go on to look at shock waves that occur in compressible flow when it goes supersonic.

9. NORMAL SHOCK WAVES

Shock waves occur in compressible fluids and are due to a sudden rise in pressure from p_1 to p_2 for example resulting from an explosion or from a sudden change in flow.

Consider a sudden rise in pressure travelling through a stream tube of fluid of constant cross sectional area A. The conditions before the change are denoted by suffix (1) and after the change by suffix (2). In particular the Mach Numbers are M_1 and M_2. We will look at the laws governing the changes one at a time starting with momentum.

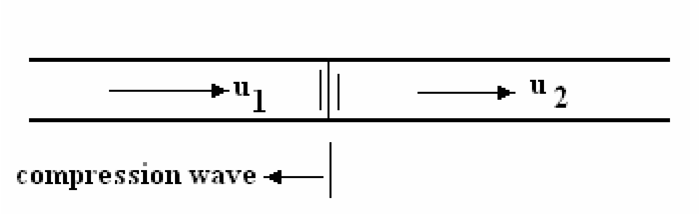

Figure 16

9.1 MOMENTUM CHANGE

From the fundamental law Force = rate of change in momentum we get

$$(p_1 - p_2) A = m (u_2 - u_1)$$

Substitute mass flow rate = m = ρA u and divide both sides by A

$$(p_1 - p_2) = (\rho_2 v_2^2 - \rho_1 v_1^2)$$

Substitute the sonic velocity $a = \sqrt{\gamma RT}$ and $RT = p/\rho$ It follows that $a^2 = \gamma p/\rho$

Mach Number is defined as $M = u/a$ so $u^2 = \gamma p M^2/\rho$

Hence $(p_1 - p_2) = (p_2\gamma M_2^2 - p_1\gamma M_1^2)$

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2} \text{........................(i)}$$

9.2 ENERGY CONSERVATION

The change in pressure is so rapid that there is no time for heat to transfer out of the gas so the pressure rise is adiabatic. In this case we may use Bernoulli.

$$c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2} \quad \text{assume } c_p \text{ is constant and substitute } u^2 = \gamma RTM^2$$

$$c_p T_1 + \frac{\gamma RT_1 M_1^2}{2} = c_p T_2 + \frac{\gamma RT_2 M_2^2}{2}$$

$$\frac{T_2}{T_1} = \frac{c_p + \frac{\gamma RM_1^2}{2}}{c_p + \frac{\gamma RM_2^2}{2}} \quad \text{substitute the relationship } R = \frac{c_p(\gamma - 1)}{\gamma} \text{ and simplify}$$

$$\frac{T_2}{T_1} = \frac{1 + (\gamma - 1)\frac{M_1^2}{2}}{1 + (\gamma - 1)\frac{M_2^2}{2}} \text{.....................(ii)}$$

9.3 CONSERVATION OF MASS

$m = \rho_1 A u_1 = \rho_2 A u_2$ so $\rho_1/\rho_2 = u_2/u_1$ but $u = a M$

$$\frac{\rho_1}{\rho_2} = \frac{a_2 M_2}{a_1 M_1} \text{ where } a = \sqrt{\gamma R T}$$

so
$$\frac{\rho_1}{\rho_2} = \frac{\sqrt{\gamma R T_2}}{\sqrt{\gamma R T_1}} \text{ x } \left(\frac{M_2}{M_1}\right) = \sqrt{\frac{T_2}{T_1}} \text{ x } \left(\frac{M_2}{M_1}\right) \text{ substitute equation (ii)}$$

$$\frac{\rho_1}{\rho_2} = \sqrt{\frac{1+(\gamma-1)\frac{M_1^2}{2}}{1+(\gamma-1)\frac{M_2^2}{2}}} \text{ x } \left(\frac{M_2}{M_1}\right) \text{................(iii)}$$

9.4 GAS LAWS

$$\frac{T_2}{T_1} = \frac{p_2 V_2}{p_1 V_1} \quad \text{so dividing top and bottom by m}$$

$$\frac{T_2}{T_1} = \frac{p_2 \rho_1}{p_1 \rho_2} \quad \text{and substituting equation (iii) gives :}$$

$$\frac{T_2}{T_1} = \sqrt{\frac{1+(\gamma-1)\frac{M_1^2}{2}}{1+(\gamma-1)\frac{M_2^2}{2}}} \text{ x } \left(\frac{M_2}{M_1}\right)\left(\frac{p_2}{p_1}\right) \text{.....................(iv)}$$

9.4 OVERALL RESULT

By combining the previous work the following equation can be obtained.

$$M_2^2 = \frac{\frac{2}{\gamma-1} + M_1^2}{\frac{2\gamma M_1^2}{\gamma-1} - 1} \text{........................(v)}$$

This equation can now be used to solve problems where the Mach number before the change is known.

Note if $M_1 = 1$ then $M_2 = 1$ and if $M_1 > 1$ then $M_2 < 1$

WORKED EXAMPLE No. 4

A gas has a temperature of 300 K, pressure of 1.5 bar and velocity of 450 m/s. Calculate the velocity, pressure and temperature after a shock wave passes into it. Take γ = 1.3 and the mean molar mass is 44.

SOLUTION

First calculate the gas constant R

$R = R_o$/molar mass = 8314/44 = 188.95 J/kg K

Note the universal gas constant R_o is on the back page of the fluids tables or should be remembered as 8314 J/kmol K

Next calculate $a_1 = \sqrt{(\gamma RT)} = \sqrt{(1.3 \times 188.95 \times 300)} = 271.46$ m/s

Hence the initial velocity is supersonic with a Mach No.

$M_1 = 450/271.46 = 1.6577$

Now calculate M_2 from equation (v)
Show for yourself that $M_2 = 0.642$

Now use equation (i) to find p_2

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}$$

Show for yourself that $p_2 = 4.47$ bar

Next use equation (ii) to find T_2

$$\frac{T_2}{T_1} = \frac{1+\frac{(\gamma-1)M_1^2}{2}}{1+\frac{(\gamma-1)M_2^2}{2}}$$

Show for yourself that $T_2 = 399$ K

Now find the sonic velocity after compression $a_2 = \sqrt{(\gamma RT)} = 313$ m/s

Hence $u_2 = a_2 M_2 = 201$ m/s

SELF ASSESSMENT EXERCISE No. 7

1. Write down the equations representing the conservation of mass, energy and momentum across a normal shock wave.

   Carbon dioxide gas enters a normal shock wave at 300 K and 1.5 bar with a velocity of 450 m/s. Calculate the pressure, temperature and velocity after the shock wave. The molecular mass is 44 kg/kmol and the adiabatic index is 1.3. (Answers 446 kPa, 399 K and 201 m/s)

2. Air discharges from a large container through a convergent - divergent nozzle into another large container at 1 bar. the exit mach number is 2.0. Determine the pressure in the container and at the throat. (Answers 7.82 bar and 4.13 bar).

   When the pressure is increased in the outlet container to 6 bar, a normal shock wave occurs in the divergent section of the nozzle. Sketch the variation of pressure, stagnation pressure, stagnation temperature and Mach number through the nozzle.

   Assume isentropic flow except through the shock. The following equations may be used.

$$\frac{\gamma RT}{\gamma-1}+\frac{u^2}{2}=\text{constant}$$

$$u = Ma\sqrt{\gamma RT}$$

$$\frac{p_1}{p_2}=\left(\frac{T_1}{T_2}\right)^{\frac{\gamma}{\gamma-1}}$$